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Partial Breaking of $\mathcal{N}=2$ Supersymmetry and of Gauge Symmetry in the $U(N)$ Gauge Model

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Abstract

We explore vacua of the $U(N)$ gauge model with $\mathcal{N}=2$ supersymmetry recently constructed in hep-th/0409060. In addition to the vacuum previously found with unbroken $U(N)$ gauge symmetry in which $\mathcal{N}=2$ supersymmetry is partially broken to $\mathcal{N}=1$, we find cases in which the gauge symmetry is broken to a product gauge group $\prod_{i=1}^{n} U(N_i)$. The $\mathcal{N}=1$ vacua are selected by the requirement of a positive definite Kähler metric. We obtain the masses of the supermultiplets appearing on the $\mathcal{N}=1$ vacua.

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I. Introduction

This is the sequel to our previous papers [1, 2] and intends to investigate further properties of our model arising from an interplay between $\mathcal{N} = 2$ supersymmetry and nonabelian gauge symmetry. In ref [1], we have successfully constructed the $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge model in four spacetime dimensions, generalizing the abelian self-interacting model given some time ago in ref [3]. The gauging of $U(N)$ isometry associated with the special Kähler geometry, and the discrete $\mathfrak{R}$ symmetry are the primary ingredients of our construction. The model spontaneously breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. The second supersymmetry realized on the broken phase acts as an approximate fermionic $U(1)$ shift symmetry. This, combined with the notion of prepotential as an input function, tells that the model should be interpreted as a low energy effective action (LEEA) designed to apply microscopic calculation invoking spectral Riemann surfaces[4, 5], matrix models (see [6] for a recent review) and/or string theory[7] to various physical processes. Although we will not investigate in this paper, it is interesting to try to find the origin of and the role played by the three parameters of our model e, m, and ξ in the developments beginning with the work of [8, 9, 10]. Connection to various compactification schemes of strings, branes and M theory[11] is anticipated and some work along this direction has already appeared [12].

Partial spontaneous breaking of extended supersymmetries appears not possible from the consideration of the algebra among the supercharges. The basic mechanism enabling the partial breaking is in fact a modification of the local version of the extended supersymmetry algebra by an additional spacetime independent term which forms a matrix with respect to extended indices [13, 14]:

$$\left\{\bar{Q}^j_{\dot{\alpha}}, \mathcal{S}^m_{\alpha i}(x)\right\} = 2(\sigma^n)_{\alpha\dot{\alpha}}\delta_i{}^j T^m_n(x) + (\sigma^m)_{\alpha\dot{\alpha}} C_i{}^j \ . \tag{1.1}$$

Note that this last term is not a vacuum expectation value but simply follows from the algebra of the extended supercurrents and from that the triplet of the auxiliary fields $\boldsymbol{D}^a$ is complex undergoing the algebraic constraints of [15].

Our model predicts

$$C_i{}^j = +4m\xi(\boldsymbol{\tau}_1)_i{}^j. \tag{1.2}$$

Separately, we find that the scalar potential takes a vacuum expectation value

$$\langle\!\langle \mathcal{V} \rangle\!\rangle = \mp 2m\xi = 2|m\xi|. \tag{1.3}$$

(See (4.17).) Once these are established, it is easy to see that partial breaking of extended supersymmetries is a reality: after ninety degree rotation, the vacuum annihilates half of the

supercharges while the remaining half takes nonvanishing and in fact infinite ($\sim |m\xi| \int d^4x$) matrix elements.

The thrust of the present paper is to explore vacua of the model in which the $U(N)$ gauge symmetry is spontaneously broken to various product gauge groups and to compute the mass spectrum on the $\mathcal{N}=1$ vacua as a function of input data, which are the prepotential derivatives and e, m and ξ. In the next section, we review the $\mathcal{N}=2$ supersymmetric $U(N)$ gauge model. Analysis of the vacua is given in section III and we determine the vacuum expectation value of the auxiliary fields $\boldsymbol{D}_a$. We also collect some properties of the derivatives of the prepotential. In section IV, we exhibit the Nambu-Goldstone fermion of the model and the $\mathcal{N}=1$ vacua are selected by the requirement of the positivity of the Kähler metric. In section V, we show that the vacua permit various breaking patterns of the $U(N)$ gauge group into product gauge groups $\prod_{i=1}^{N} U(N_i)$. The "triplet-doublet splitting" at $N=5$ is discussed. Finally in section VI, we compute the masses of the bosons and of the fermions of the model and obtain three types of $\mathcal{N}=1$ (on-shell) supermultiplets. In the Appendix, we collect some formulas associated with the standard basis of the $u(N)$ Lie algebra.

It will be appropriate to introduce here notation to label the generators of $u(N)$ Lie algebra by indices. A set of $u(N)$ generators is first labelled by indices $a, b, ... = 0, 1, ... N^2-1$. Here 0 refers to the overall $U(1)$ generator. In a basis in which the decoupling of $U(1)$ is manifest, we label the generators belonging to the maximal Cartan subalgebra by $i, j, k, ...$ while the generators belonging to the roots (the non-Cartan generators) are labelled by $r, s, ...$. In our analysis of vacua, we employ the standard basis of the $u(N)$ Lie algebra. See the Appendix for this basis. The diagonal generators in this basis are labelled by $\underline{i}, \underline{j}, \underline{k}, ...$ and are referred to as those in the eigenvalue basis. The non-Cartan generators associated with unbroken gauge symmetry are labelled by $r', s', ...$ while the remaining non-Cartan generators representing broken gauge symmetry are labelled by $\mu, \nu, ...$. Our physics output, mass spectrum of our model only distinguishes the broken generators from the unbroken ones. We therefore introduce $\alpha, \beta, \gamma, ...$ as a union of $\underline{i}, \underline{j}, \underline{k}, ...$ and $r, s, ...$ in order to label the entire unbroken generators. Our final formula will be expressible in terms of $\alpha, \beta, ...$ and $\mu, \nu, ...$ only.

II. Review of the $U(N)$ gauge model

The $\mathcal{N}=2$ $U(N)$ gauge model constructed in [1] is composed of a set of $\mathcal{N}=1$ chiral multiplets $\Phi = \Phi^a t_a$ and a set of $\mathcal{N}=1$ vector multiplets $V = V^a t_a$, where $N \times N$ hermitian matrices t_a, $(a = 0, \ldots N^2-1)$, generate $u(N)$, $[t_a, t_b] = i f_{ab}^c t_c$. These superfields, Φ^a and V^a,

contain component fields (A^a, ψ^a, F^a) and (v_m^a, λ^a, D^a), respectively. This model is described by an analytic function $\mathcal{F}(\Phi)$.[♭] The kinetic term of Φ is given by the Kähler potential $K(\Phi^a, \Phi^{*a}) = \frac{i}{2}(\Phi^a \mathcal{F}_a^* - \Phi^{*a}\mathcal{F}_a)$ of the special Kähler geometry as $\mathcal{L}_K = \int d^2\theta^2 d\bar{\theta}^2 K(\Phi^a, \Phi^{*a})$. The Kähler metric $g_{ab^*} \equiv \partial_a \partial_{b^*} K(A^a, A^{*a}) = \mathrm{Im}\, \mathcal{F}_{ab}$ admits isometry $U(N)$. The $U(N)$ gauging is accomplished [16, 17] by adding to $\mathcal{L}_K$ $\mathcal{L}_\Gamma$ which is specified by the Killing potential $\mathfrak{D}_a = -i g_{ab} f_{cd}^b A^{*c} A^d$. The kinetic term of V is given as $\mathcal{L}_{\mathcal{W}^2} = -\frac{i}{4}\int d^2\theta^2 \mathcal{F}_{ab} \mathcal{W}^a \mathcal{W}^b + c.c$, where $\mathcal{W}^a$ is the gauge field strength of V^a. This model contains the superpotential term $\mathcal{L}_W = \int d\theta^2 W + c.c$ with $W = eA^0 + m\mathcal{F}_0$, and the Fayet-Iliopoulos D-term $\mathcal{L}_D = \sqrt{2}\xi D^0$ as well [18] . Gathering these together, the total Lagrangian of the $\mathcal{N} = 2$ $U(N)$ model is given as

$$\mathcal{L} \;=\; \mathcal{L}_K + \mathcal{L}_\Gamma + \mathcal{L}_{\mathcal{W}^2} + \mathcal{L}_W + \mathcal{L}_D \;. \tag{2.1}$$

Eliminating the auxiliary fields by using their equations of motion

$$D^a \;=\; \hat{D}^a - \frac{1}{2} g^{ab}\left(\mathfrak{D}_b + 2\sqrt{2}\xi\delta_b^0\right) \;, \quad \hat{D}^a \equiv -\frac{\sqrt{2}}{4} g^{ab}\left(\mathcal{F}_{bcd}\psi^d\lambda^c + \mathcal{F}_{bcd}^*\bar{\psi}^d\bar{\lambda}^c\right) \;, \tag{2.2}$$

$$F^a \;=\; \hat{F}^a - g^{ab^*}\partial_{b^*} W^* \;, \quad \hat{F}^a \equiv \frac{i}{4} g^{ab^*}\left(\mathcal{F}_{bcd}^*\bar{\lambda}^c\bar{\lambda}^d - \mathcal{F}_{bcd}\psi^c\psi^d\right) \;, \tag{2.3}$$

the Lagrangian $\mathcal{L}$ (2.1) takes the following form:

$$\mathcal{L}' \;=\; \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{pot}} + \mathcal{L}_{\mathrm{Pauli}} + \mathcal{L}_{\mathrm{mass}} + \mathcal{L}_{\mathrm{fermi}^4} \;, \tag{2.4}$$

with

$$\begin{aligned}
\mathcal{L}_{\mathrm{kin}} \;=\;& -g_{ab^*}\mathcal{D}_m A^a \mathcal{D}^m A^{*b} - \frac{1}{4} g_{ab} v_{mn}^a v^{bmn} - \frac{1}{8}\mathrm{Re}(\mathcal{F}_{ab})\,\epsilon^{mnpq} v_{mn}^a v_{pq}^b \\
& + \left[-\frac{1}{2}\mathcal{F}_{ab}\lambda^a\sigma^m\mathcal{D}_m\bar{\lambda}^b - \frac{1}{2}\mathcal{F}_{ab}\psi^a\sigma^m\mathcal{D}_m\bar{\psi}^b \;+\; c.c.\right] \;,
\end{aligned} \tag{2.5}$$

$$\mathcal{L}_{\mathrm{pot}} \;=\; -g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \xi^2\delta_a^0\delta_b^0\right) - g^{ab^*}\partial_a W \partial_{b^*} W^*, \tag{2.6}$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{mass}} \;=\;& \left[-\frac{i}{4} g^{cd^*}\mathcal{F}_{abc}\partial_{d^*} W^*(\psi^a\psi^b + \lambda^a\lambda^b) \;+ c.c.\right] \\
& + \left[\frac{1}{2\sqrt{2}}\left(g_{ac^*}k_b^{*c} - g_{bc^*}k_a^{*c} - \sqrt{2}\xi\delta_c^0 g^{cd}\mathcal{F}_{abd}\right)\psi^a\lambda^b \;+ c.c.\right] \;,
\end{aligned} \tag{2.7}$$

where we have defined the covariant derivative by $\mathcal{D}_m\Psi^a \equiv \partial_m\Psi^a - \frac{1}{2} f_{bc}^a v_m^b \Psi^c$ for $\Psi^a \in \{A^a, \psi^a, \lambda^a\}$, and $v_{mn}^a \equiv \partial_m v_n^a - \partial_n v_m^a - \frac{1}{2} f_{bc}^a v_m^b v_n^c$. The holomorphic Killing vectors $k_a = k_a{}^b\partial_b$ are generated by the Killing potential $\mathfrak{D}_a$ as $k_a{}^b = -i g^{bc^*}\partial_{c^*}\mathfrak{D}_a$ and $k_a^{*b} = i g^{b^*c}\partial_c\mathfrak{D}_a$. Here, we have omitted $\mathcal{L}_{\mathrm{Pauli}}$ and $\mathcal{L}_{\mathrm{fermi}^4}$ as they are irrelevant for our purposes in this paper.

[♭] $\mathcal{F}_a \equiv \partial_a\mathcal{F}$ and $\mathcal{F}_{ab} \equiv \partial_a\partial_b\mathcal{F}$

In [1], Lagrangians, $\mathcal{L}$ and $\mathcal{L}'$, are shown to be invariant under the $\mathfrak{R}$-action,

$$\mathfrak{R}: \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix} \to \begin{pmatrix} \psi^a \\ -\lambda^a \end{pmatrix}, \quad \xi \to -\xi \quad , \tag{2.8}$$

and for $\mathcal{L}'$ in addition

$$\mathfrak{R}: \begin{array}{c} F^a + g^{ac^*}\partial_{c^*}W^* \to F^{*b} + g^{db^*}\partial_d W \ , \\ D^c + \frac{1}{2}g^{cd}\mathfrak{D}_d \to -(D^c + \frac{1}{2}g^{cd}\mathfrak{D}_d) \ . \end{array} \tag{2.9}$$

This property guarantees the $\mathcal{N}=2$ supersymmetry of the model as follows. By construction, the action, $S \equiv \int \mathcal{L}$ or $\int \mathcal{L}'$, is invariant under the $\mathcal{N}=1$ supersymmetry, $\delta_1 S = 0$. Operating the $\mathfrak{R}$-action on this equation, one finds $0 = \mathfrak{R}\delta_1 S \mathfrak{R}^{-1} = \mathfrak{R}\delta_1\mathfrak{R}^{-1}S$. This implies that S is invariant under the second supersymmetry $\delta_2 \equiv \mathfrak{R}\delta_1\mathfrak{R}^{-1}$ as well in addition to the first supersymmetry δ_1. We have also given in [1] (see Appendix A) a proof of $\mathcal{N}=2$ supersymmetry of our action, using the canonical transformation acting only on the fields without invoking $\xi \to -\xi$. The $\mathcal{N}=2$ supersymmetry transformations are obtained by covariantizing the $\mathcal{N}=1$ transformations with respect to $\mathfrak{R}$. Using the doublet of fermions

$$\boldsymbol{\lambda}_I{}^a \equiv \begin{pmatrix} \lambda^a \\ \psi^a \end{pmatrix}, \quad \boldsymbol{\lambda}^{Ia} \equiv \epsilon^{IJ}\boldsymbol{\lambda}_J{}^a \ , \quad \bar{\boldsymbol{\lambda}}^{Ia} \equiv \begin{pmatrix} \bar{\lambda}^a \\ \bar{\psi}^a \end{pmatrix}, \quad \bar{\boldsymbol{\lambda}}_I{}^a \equiv \epsilon_{IJ}\bar{\boldsymbol{\lambda}}^{Ja} \ , \tag{2.10}$$

and the doublet of supersymmetry transformation parameters

$$\boldsymbol{\eta}_I \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \boldsymbol{\eta}^I \equiv \epsilon^{IJ}\boldsymbol{\eta}_J \ , \quad \bar{\boldsymbol{\eta}}^J \equiv \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}, \quad \bar{\boldsymbol{\eta}}_J \equiv \epsilon_{JI}\bar{\boldsymbol{\eta}}^I \ , \tag{2.11}$$

where ϵ^{IJ} is given by $\epsilon^{12} = \epsilon_{21} = 1$ and $\epsilon^{21} = \epsilon_{12} = -1$, the $\mathcal{N}=2$ transformations are written as

$$\boldsymbol{\delta} A^a = \sqrt{2}\boldsymbol{\eta}_J\boldsymbol{\lambda}^{Ja}, \tag{2.12}$$

$$\boldsymbol{\delta} v_m^a = i\boldsymbol{\eta}_J\sigma_m\bar{\boldsymbol{\lambda}}^{Ja} - i\boldsymbol{\lambda}_J{}^a\sigma_m\bar{\boldsymbol{\eta}}^J \ , \tag{2.13}$$

$$\boldsymbol{\delta}\boldsymbol{\lambda}_J{}^a = (\sigma^{mn}\boldsymbol{\eta}_J)v_{mn}^a + \sqrt{2}i(\sigma^m\bar{\boldsymbol{\eta}}_J)\mathcal{D}_m A^a + i(\boldsymbol{\tau}\cdot\boldsymbol{D}^a)_J{}^K\boldsymbol{\eta}_K - \frac{1}{2}\boldsymbol{\eta}_J f^a_{\ bc}A^{*b}A^c \ . \tag{2.14}$$

Here, $\boldsymbol{D}^a$ represent the three-vectors

$$\boldsymbol{D}^a = \hat{\boldsymbol{D}}^a - \sqrt{2}g^{ab^*}\partial_{b^*}\left(\boldsymbol{\mathcal{E}}A^{*0} + \boldsymbol{\mathcal{M}}\mathcal{F}_0^*\right), \tag{2.15}$$

$$\hat{\boldsymbol{D}}^a = (\sqrt{2}\,\mathrm{Im}\,\hat{F}^a, -\sqrt{2}\,\mathrm{Re}\,\hat{F}^a, \hat{D}^a), \tag{2.16}$$

$$\boldsymbol{\mathcal{E}} = (0, \ -e, \ \xi) \ , \quad \boldsymbol{\mathcal{M}} = (0, \ -m, \ 0), \tag{2.17}$$

and $\boldsymbol{\tau}$ are the Pauli matrices. Let us also note that

$$\mathrm{Im}\,\boldsymbol{D}^a = \delta^a{}_0(-\sqrt{2}m)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} . \tag{2.18}$$

This simply follows from (3.4) and also a consequence from the superspace constraints derived in [15].

The construction of the extended supercurrents is somewhat involved as is fully discussed in [1] (See [19, 20] for this). Nonetheless it is easy to extract the piece contributing to $C_i{}^j$ in the algebra (1.1). It comes from the structure of

$$\boldsymbol{\tau}\cdot\boldsymbol{D}^{*b}\boldsymbol{\tau}\cdot\boldsymbol{D}^a = \boldsymbol{D}^{*b}\cdot\boldsymbol{D}^a\mathbf{1} + i\left(\boldsymbol{D}^{*b}\times\boldsymbol{D}^a\right)\cdot\boldsymbol{\tau}. \tag{2.19}$$

The second term is nonvanishing only for complex $\boldsymbol{D}^a$ and this is the piece responsible for $C_i{}^j$. After using (2.15) and (2.18), we derive (1.2).

III. Analysis of vacua

Let us examine the scalar potential of the model, $\mathcal{V} = -\mathcal{L}_{\mathrm{pot}}$,

$$\begin{aligned} \mathcal{V} &= g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \xi^2\delta_a^0\delta_b^0 + \partial_a W\partial_{b^*}W^*\right) \\ &= g^{ab}\left(\frac{1}{8}\mathfrak{D}_a\mathfrak{D}_b + \partial_a\left(\boldsymbol{\mathcal{E}}A^0 + \boldsymbol{\mathcal{M}}\mathcal{F}_0\right)\cdot\partial_{b^*}\left(\boldsymbol{\mathcal{E}}A^0 + \boldsymbol{\mathcal{M}}\mathcal{F}_0\right)^*\right) . \end{aligned} \tag{3.1}$$

For our present purpose, it is more useful to convert this expression into

$$\mathcal{V} = \frac{1}{8}g_{bc}\mathfrak{D}^b\mathfrak{D}^c + \frac{1}{2}g^{bc}\tilde{\boldsymbol{D}}_b^*\cdot\tilde{\boldsymbol{D}}_c\ , \tag{3.2}$$

where

$$\mathfrak{D}^a = g^{ab}\mathfrak{D}_b = -if^a_{cd}A^{*c}A^d\ , \tag{3.3}$$

and

$$\tilde{\boldsymbol{D}}_b \equiv g_{ba}\tilde{\boldsymbol{D}}^a = -\sqrt{2}\partial_{b^*}\left(\boldsymbol{\mathcal{E}}A^{*0} + \boldsymbol{\mathcal{M}}\mathcal{F}_0^*\right) = \sqrt{2}\begin{pmatrix} 0 \\ \partial_{b^*}W^* \\ -\xi\delta_b{}^0 \end{pmatrix} , \tag{3.4}$$

as well as (2.18). We derive from (3.2)-(3.4),

$$\begin{aligned} \frac{\partial\mathcal{V}}{\partial A^a} &= \frac{1}{4}g_{bc}\partial_a\mathfrak{D}^b\mathfrak{D}^c + \frac{1}{8}\partial_a g_{bc}\mathfrak{D}^b\mathfrak{D}^c + \frac{i}{4}\mathcal{F}_{abc}\tilde{\boldsymbol{D}}^{*b}\cdot\tilde{\boldsymbol{D}}^c - \frac{\sqrt{2}}{2}\mathcal{F}_{abc}\boldsymbol{\mathcal{M}}\delta_0^b\cdot\tilde{\boldsymbol{D}}^c \\ &= \frac{1}{4}g_{bc}\partial_a\mathfrak{D}^b\mathfrak{D}^c + \frac{1}{8}\partial_a g_{bc}\mathfrak{D}^b\mathfrak{D}^c + \frac{i}{4}\mathcal{F}_{abc}\tilde{\boldsymbol{D}}^b\cdot\tilde{\boldsymbol{D}}^c. \end{aligned} \tag{3.5}$$

This is an expression valid in any vacuum and generalizes the one found in [1] (eq.(4.17)) for the unbroken vacuum.

In order to examine general vacua, let indices $a=(i,r)$ and $i\,(r)$ label the (non) Cartan generators of $u(N)$ (see the Appendix). We are interested in the vacuum at which $\langle A^r\rangle = 0$. The vacuum condition (3.5) reduces [2] to

$$0 = \frac{i}{4}\langle \mathcal{F}_{abc}\boldsymbol{D}^b\cdot\boldsymbol{D}^c\rangle, \tag{3.6}$$

because $\langle \mathfrak{D}^a\rangle = \langle -if^a_{ij}A^{*i}A^j\rangle = 0$. Here and henceforth, we drop the tilde on $\boldsymbol{D}^a$ noting that $\langle \tilde{\boldsymbol{D}}^a\rangle = \langle \boldsymbol{D}^a\rangle$ as the vacuum expectation value.

It is more convenient to work on the set of bases (the eigenvalue bases) in which the Cartan subalgebra of $u(N)$ is spanned by

$$(t_{\underline{i}})_j{}^k = \delta_{\underline{i}}{}^k\delta_j{}^{\underline{i}}\ . \qquad (\underline{i}=1\sim N\ , \quad j,k=1\sim N\ .) \tag{3.7}$$

These $t_{\underline{i}}$ correspond to $H_{\underline{i}}$ in the Appendix. Let us introduce a matrix which transforms the standard bases labelled by i,j,k of the Cartan generators into the eigenvalue bases labelled by $\underline{i},\underline{j},\underline{k}$ as

$$t_{\underline{i}} = O_{\underline{i}}{}^j t_j\ , \tag{3.8}$$

$$t_i = O_i{}^{\underline{j}} t_{\underline{j}}\ . \tag{3.9}$$

This matrix satisfies $O_i{}^{\underline{j}}O_{\underline{j}}{}^k = \delta_i{}^k$ and $O_{\underline{i}}{}^jO_j{}^{\underline{k}} = \delta_{\underline{i}}{}^{\underline{k}}$. We normalize the standard $u(N)$ Cartan generators t_i as $\mathrm{tr}(t_it_j) = \frac{1}{2}\delta_{ij}$, which implies that the overall $u(1)$ generator is $t_0 = \frac{1}{\sqrt{2N}}\mathbf{1}_{N\times N}$. Let us derive useful relations which will be exploited in what follows. Summing up equation (3.8) with respect to $\underline{i}$, we find $\sqrt{2N}t_0 = \sum_{\underline{j}} O_{\underline{j}}{}^i t_i$, and thus $\sum_{\underline{j}} O_{\underline{j}}{}^i = \sqrt{2N}\delta_0{}^i$. Taking the trace of (3.8), we obtain

$$O_{\underline{i}}{}^0 = \sqrt{\frac{2}{N}}\ . \tag{3.10}$$

Taking the trace of (3.9), we obtain $\sqrt{\frac{N}{2}}\delta_i{}^0 = \sum_{\underline{j}} O_i{}^{\underline{j}}$. It follows from (3.9) with $i=0$ that

$$O_0{}^{\underline{j}} = \frac{1}{\sqrt{2N}}\ . \tag{3.11}$$

On the other hand, non-Cartan generators t_r are $E^{\pm}_{\underline{ij}} = \pm E^{\pm}_{\underline{ji}}$ in the Appendix. Thus Φ can be expanded as $\Phi = \sum_i \Phi^i t_i + \sum_r \Phi^r t_r = \sum_{\underline{i}} \Phi^{\underline{i}} t_{\underline{i}} + \frac{1}{2}\sum_{\underline{i},\underline{j}(\underline{i}\neq\underline{j})}\left(\Phi_+^{\underline{ij}}E^+_{\underline{ij}} + \Phi_-^{\underline{ij}}E^-_{\underline{ij}}\right)$ with $\Phi_\pm^{\underline{ij}} = \pm\Phi_\pm^{\underline{ji}}$. In the Appendix, we explain our notation in some detail.

Let us collect some properties of $\langle\mathcal{F}_{ab}\rangle$ and $\langle\mathcal{F}_{abc}\rangle$ for the following $\mathcal{F}$:

$$\mathcal{F}=\sum_{\ell=0}^{k}\frac{C_\ell}{\ell!}\,\mathrm{tr}\,\Phi^\ell. \tag{3.12}$$

We note that the matrix $\langle\Phi\rangle$ is complex and normal and the eigenvalues $\lambda^{\underline{i}}$ are in general complex. Letting $\langle\Phi\rangle=\lambda^{\underline{i}}t_{\underline{i}}$, we find that the nonvanishing $\langle\mathcal{F}_{ab}\rangle$ are

$$\begin{aligned}\langle\mathcal{F}_{\underline{ii}}\rangle &= \sum_\ell \frac{C_\ell}{(\ell-2)!}(\lambda^{\underline{i}})^{\ell-2}\ , & (3.13)\\ \langle\mathcal{F}_{\pm\underline{ij},\pm\underline{ij}}\rangle &\equiv \langle\frac{\partial^2\mathcal{F}}{\partial\Phi_\pm^{\underline{ij}}\partial\Phi_\pm^{\underline{ij}}}\rangle=\begin{cases}\displaystyle\sum_\ell\frac{C_\ell}{2(\ell-1)!}\frac{(\lambda^{\underline{i}})^{\ell-1}-(\lambda^{\underline{j}})^{\ell-1}}{\lambda^{\underline{i}}-\lambda^{\underline{j}}} & \text{if }\ \lambda^{\underline{i}}\neq\lambda^{\underline{j}}\ ,\\ \displaystyle\sum_\ell\frac{C_\ell}{2(\ell-2)!}(\lambda^{\underline{i}})^{\ell-2} & \text{if }\ \lambda^{\underline{i}}=\lambda^{\underline{j}}\ .\end{cases} & (3.14)\end{aligned}$$

See the Appendix . These imply that $\langle g_{ab}\rangle$ is diagonal: $\langle g_{\underline{ij}}\rangle\propto\delta_{\underline{ij}}$, $\langle g_{rs}\rangle\propto\delta_{rs}$ and $\langle g_{ir}\rangle=\langle g^{ir}\rangle=0$. In addition, we have $\langle\mathcal{F}_{+\underline{ij},+\underline{ij}}\rangle=\langle\mathcal{F}_{-\underline{ij},-\underline{ij}}\rangle$, i.e. $\langle g_{+\underline{ij},+\underline{ij}}\rangle=\langle g_{-\underline{ij},-\underline{ij}}\rangle$. In particular we note that, for directions $\Phi_\pm^{\underline{ij}}$ with $\lambda^{\underline{i}}=\lambda^{\underline{j}}$, $\langle\mathcal{F}_{\pm\underline{ij},\pm\underline{ij}}\rangle=\frac{1}{2}\langle\mathcal{F}_{\underline{ii}}\rangle$, i.e. $\langle g_{\pm\underline{ij},\pm\underline{ij}}\rangle=\frac{1}{2}\langle g_{\underline{ii}}\rangle$. For $\langle\mathcal{F}_{abc}\rangle$, the nonvanishing components are

$$\begin{aligned}\langle\mathcal{F}_{\underline{iii}}\rangle &= \sum_\ell\frac{C_\ell}{(\ell-3)!}(\lambda^{\underline{i}})^{\ell-3}\ , & (3.15)\\ \langle\mathcal{F}_{\underline{k},\pm\underline{ij},\pm\underline{ij}}\rangle &= \begin{cases}\displaystyle\sum_\ell\frac{C_\ell}{2(\ell-1)!}(\delta_{\underline{ik}}\frac{\partial}{\partial\lambda^{\underline{i}}}+\delta_{\underline{jk}}\frac{\partial}{\partial\lambda^{\underline{j}}})\frac{(\lambda^{\underline{i}})^{\ell-1}-(\lambda^{\underline{j}})^{\ell-1}}{\lambda^{\underline{i}}-\lambda^{\underline{j}}} & \text{if }\ \lambda^{\underline{i}}\neq\lambda^{\underline{j}}\ ,\\ \displaystyle\sum_\ell\frac{C_\ell}{2(\ell-3)!}\delta_{\underline{ik}}(\lambda^{\underline{i}})^{\ell-3} & \text{if }\ \lambda^{\underline{i}}=\lambda^{\underline{j}}\ .\end{cases} & (3.16)\end{aligned}$$

Obviously $\partial_{\underline{i}}\langle\mathcal{F}_{abc\cdots}\rangle=\langle\partial_{\underline{i}}\mathcal{F}_{abc\cdots}\rangle$. Finally, we note here remarkable relations

$$\langle\mathcal{F}_{0,\pm\underline{ij},\pm\underline{ij}}\rangle=\begin{cases}\displaystyle\sum_\ell\frac{C_\ell}{2\sqrt{2N}(\ell-2)!}\frac{(\lambda^{\underline{i}})^{\ell-2}-(\lambda^{\underline{j}})^{\ell-2}}{\lambda^{\underline{i}}-\lambda^{\underline{j}}}=\frac{\langle\mathcal{F}_{\underline{ii}}\rangle-\langle\mathcal{F}_{\underline{jj}}\rangle}{2\sqrt{2N}(\lambda^{\underline{i}}-\lambda^{\underline{j}})} & \text{if }\ \lambda^{\underline{i}}\neq\lambda^{\underline{j}}\ ,\\ \displaystyle\frac{1}{2\sqrt{2N}}\langle\mathcal{F}_{\underline{iii}}\rangle & \text{if }\ \lambda^{\underline{i}}=\lambda^{\underline{j}}\ ,\end{cases} \tag{3.17}$$

which play a key role in the analysis of the mass spectrum.

Let us return to the vacuum condition (3.6). This condition is automatically satisfied for $a=r$ because $\langle\boldsymbol{D}^r\rangle=-\sqrt{2}\langle g^{rs}(\boldsymbol{\mathcal{E}}\delta_s^0+\boldsymbol{\mathcal{M}}\mathcal{F}_{0s}^*)\rangle=0$. Noting that the only nonvanishing second and third derivatives for the Cartan directions are the diagonal ones, namely, $\mathcal{F}_{\underline{ii}}$ and $\mathcal{F}_{\underline{iii}}$, the vacuum condition (3.6) reduces to

$$\left\langle\mathcal{F}_{\underline{jjj}}\boldsymbol{D}^{\underline{j}}\cdot\boldsymbol{D}^{\underline{j}}\right\rangle=0 \qquad \text{with } \underline{j} \text{ not summed},\quad 1\leq\underline{j}\leq N\ . \tag{3.18}$$

The points specified by $\langle \mathcal{F}_{\underline{jjj}} \rangle = 0$ are not stable vacua because $\langle \partial_{\underline{j}} \partial_{\underline{j}^*} \mathcal{V} \rangle = 0$. At the stable vacua, we obtain $\left\langle \boldsymbol{D}^{\underline{j}} \cdot \boldsymbol{D}^{\underline{j}} \right\rangle = 0$, or equivalently

$$\left\langle \boldsymbol{D}_{\underline{j}} \cdot \boldsymbol{D}_{\underline{j}} \right\rangle = 0 \ , \tag{3.19}$$

where

$$\langle \boldsymbol{D}_{\underline{i}} \rangle = O_{\underline{i}}{}^{j} \langle \boldsymbol{D}_j \rangle = \sqrt{2} \begin{pmatrix} 0 \\ \langle \frac{\partial}{\partial \lambda^{*\underline{i}}} W^* \rangle \\ \langle -\xi \frac{\partial}{\partial \lambda^{*\underline{i}}} (A^{*0}) \rangle \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ e O_{\underline{i}}{}^{0} + m O_0{}^{\underline{i}} \langle \mathcal{F}^*_{\underline{ii}} \rangle \\ -\xi O_{\underline{i}}{}^{0} \end{pmatrix} . \tag{3.20}$$

We have determined the vacuum expectation values of the following quantities:

$$m \langle\!\langle \mathcal{F}^*_{\underline{jj}} \rangle\!\rangle \ = \ -\frac{O_{\underline{j}}{}^{0}}{O_0{}^{\underline{j}}} (e \mp i\xi) = -2(e \mp i\xi), \tag{3.21}$$

$$\langle\!\langle g_{\underline{jj}} \rangle\!\rangle \ = \ \mp 2 \frac{\xi}{m}, \tag{3.22}$$

$$\langle\!\langle \boldsymbol{D}_{\underline{j}} \rangle\!\rangle \ = \ 2 \frac{\xi}{\sqrt{N}} \begin{pmatrix} 0 \\ \pm i \\ -1 \end{pmatrix} \equiv \boldsymbol{d}_{(\pm)}, \tag{3.23}$$

$$\langle\!\langle \boldsymbol{D}^{\underline{j}} \rangle\!\rangle \ = \ \frac{m}{\sqrt{N}} \begin{pmatrix} 0 \\ -i \\ \pm 1 \end{pmatrix} \equiv \boldsymbol{d}^{(\pm)}. \tag{3.24}$$

We use $\langle\!\langle \cdots \rangle\!\rangle$ for those vacuum expectation values which are determined as the solutions to (3.18). Note that the sign factor $\pm$ can be chosen freely for each $\underline{j}$. Let $\boldsymbol{\mathcal{M}}_+$ be the set chosen from $1, \ldots, N$ in which the $+$ sign is chosen in (3.24) and let $\boldsymbol{\mathcal{M}}_-$ be the one in which the $-$ sign is chosen. We will see shortly that this determines the number of spontaneously broken supersymmetries. As one sees from (2.5), $\langle\!\langle \mathrm{Im}\, \mathcal{F}_{ab} \rangle\!\rangle$ measures the inverse of the squared coupling constant $\frac{1}{g_{YM}^2}$, while $\langle\!\langle \mathrm{Re}\, \mathcal{F}_{ab} \rangle\!\rangle$ is the θ-angle of QCD.

IV. Partially broken supersymmetry and NG fermions.

Let us examine the supersymmetry transformation of fermions (2.14), which reduces at the vacuum to

$$\langle\!\langle \boldsymbol{\delta \lambda}^a_I \rangle\!\rangle = i \langle\!\langle (\boldsymbol{\tau} \cdot \boldsymbol{D}^a)_I{}^J \rangle\!\rangle \boldsymbol{\eta}_J \ . \tag{4.1}$$

As $\langle\!\langle \boldsymbol{D}^r \rangle\!\rangle = 0$, the fermion $\boldsymbol{\lambda}^r_I$ on the vacuum is invariant under the supersymmetry

$$\langle\!\langle \boldsymbol{\delta \lambda}^r_I \rangle\!\rangle = 0 \ . \tag{4.2}$$

The Nambu-Goldstone fermion signaling supersymmetry breaking is contained in $\boldsymbol{\lambda}_I^{\underline{i}}$. For $\boldsymbol{\delta\lambda}_I^{\underline{i}}$, the 2×2 matrix $\boldsymbol{\tau}\cdot\boldsymbol{D}^{\underline{i}}$ is easily diagonalized as

$$\langle\!\langle \delta\left(\frac{\lambda^{\underline{i}}\pm\psi^{\underline{i}}}{\sqrt{2}}\right)\rangle\!\rangle = \mp\frac{1}{\sqrt{2}}\langle\!\langle D_2^{\underline{i}}\mp iD_3^{\underline{i}}\rangle\!\rangle(\eta_1\mp\eta_2)\ . \tag{4.3}$$

If either $\boldsymbol{\mathcal{M}}_-$ or $\boldsymbol{\mathcal{M}}_+$ is empty, $\langle\!\langle\boldsymbol{D}^{\underline{j}}\rangle\!\rangle$ ($\underline{j}=1,...,N$) we have obtained are all identical. When $\boldsymbol{\mathcal{M}}_-$ is empty, the matrix $\boldsymbol{\tau}\cdot\boldsymbol{D}^{\underline{j}}=\boldsymbol{\tau}\cdot\boldsymbol{d}^{(+)}$ is of rank 1, which signals the partial supersymmetry breaking:

$$\langle\!\langle \delta\left(\frac{\lambda^{\underline{i}}+\psi^{\underline{i}}}{\sqrt{2}}\right)\rangle\!\rangle = im\sqrt{\frac{2}{N}}(\eta_1-\eta_2)\ , \tag{4.4}$$

$$\langle\!\langle \delta\left(\frac{\lambda^{\underline{i}}-\psi^{\underline{i}}}{\sqrt{2}}\right)\rangle\!\rangle = 0\ . \tag{4.5}$$

In the original Cartan basis, this means that $\langle\!\langle\delta\left(\frac{\lambda^i+\psi^i}{\sqrt{2}}\right)\rangle\!\rangle = 2im\delta_0^i(\eta_1-\eta_2)$ where we have used the fact $\sum_{\underline{j}} O_{\underline{j}}{}^i=\sqrt{2N}\delta_0^i$. As we will show in section VI, the fermion $\frac{1}{\sqrt{2}}(\lambda^i+\psi^i)$ are massless while $\frac{1}{\sqrt{2}}(\lambda^i-\psi^i)$ are massive. Thus, $\mathcal{N}=2$ supersymmetry is spontaneously broken to $\mathcal{N}=1$ and we obtain the Nambu-Goldstone fermion $\frac{1}{\sqrt{2}}(\lambda^0+\psi^0)$ associated with the overall $U(1)$ part. The same reasoning holds when $\boldsymbol{\mathcal{M}}_+$ is empty. On the other hand, if neither $\boldsymbol{\mathcal{M}}_+$ nor $\boldsymbol{\mathcal{M}}_-$ is empty, we have two independent rank one matrices $\boldsymbol{\tau}\cdot\boldsymbol{d}^{(+)}$ and $\boldsymbol{\tau}\cdot\boldsymbol{d}^{(-)}$ and $\mathcal{N}=2$ supersymmetry is spontaneously broken to $\mathcal{N}=0$. Which part of the Cartan subalgebra of $u(N)$ contains two Nambu-Goldstone fermions depend upon the type of grouping of $1\sim N$ into $\boldsymbol{\mathcal{M}}_+$ and $\boldsymbol{\mathcal{M}}_-$. Let $\underline{i}=(\underline{i}',\underline{i}'')$ with $\underline{i}'\in\boldsymbol{\mathcal{M}}_+$ and $\underline{i}''\in\boldsymbol{\mathcal{M}}_-$, then

$$\langle\!\langle \delta\left(\frac{\lambda^{\underline{i}'}+\psi^{\underline{i}'}}{\sqrt{2}}\right)\rangle\!\rangle = im\sqrt{\frac{2}{N}}(\eta_1-\eta_2)\ ,\quad \langle\!\langle \delta\left(\frac{\lambda^{\underline{i}''}+\psi^{\underline{i}''}}{\sqrt{2}}\right)\rangle\!\rangle = 0\ , \tag{4.6}$$

$$\langle\!\langle \delta\left(\frac{\lambda^{\underline{i}''}-\psi^{\underline{i}''}}{\sqrt{2}}\right)\rangle\!\rangle = -im\sqrt{\frac{2}{N}}(\eta_1+\eta_2)\ ,\quad \langle\!\langle \delta\left(\frac{\lambda^{\underline{i}'}-\psi^{\underline{i}'}}{\sqrt{2}}\right)\rangle\!\rangle = 0\ . \tag{4.7}$$

As is obvious from the similar analysis given in section VI, the fermions, $\frac{1}{\sqrt{2}}(\lambda^{\underline{i}'}+\psi^{\underline{i}'})$ and $\frac{1}{\sqrt{2}}(\lambda^{\underline{i}''}-\psi^{\underline{i}''})$, are massless and contain two Nambu-Goldstone fermions of $\mathcal{N}=2$ supersymmetry broken to $\mathcal{N}=0$.

We comment on the vacuum value of the scalar potential $\mathcal{V}$. For the $\mathcal{N}=1$ vacua, we have

$$\langle\!\langle\mathcal{V}\rangle\!\rangle = \mp 2m\xi, \tag{4.8}$$

where the $\mp$ signs correspond respectively to the cases $\forall\underline{i}\in\boldsymbol{\mathcal{M}}_\pm$. For the $\mathcal{N}=0$ vacua, on the other hand, we have

$$\langle\!\langle\mathcal{V}\rangle\!\rangle = -2m\xi\frac{1}{N}\left(\mathrm{ord}(\boldsymbol{\mathcal{M}}_+)-\mathrm{ord}(\boldsymbol{\mathcal{M}}_-)\right)\ , \tag{4.9}$$

where $\mathrm{ord}(\boldsymbol{\mathcal{M}}_{\pm})$ is the number of elements of $\boldsymbol{\mathcal{M}}_{\pm}$.

We now impose the positivity criterion of the Kähler metric to select the physical vacua. For the $\mathcal{N}=1$ vacua, this criterion requires

$$\langle\!\langle g_{\underline{ii}} \rangle\!\rangle = \mp 2\frac{\xi}{m} > 0. \tag{4.10}$$

Depending upon $\frac{\xi}{m} \lessgtr 0$, we must choose either one of the two possibilities discussed above. For the $\mathcal{N}=0$ vacua, however, the Kähler metric cannot be positive definite, causing unconventional signs for the kinetic term. The $\mathcal{N}=0$ vacua are regarded as unphysical.

In the subsequent sections, we will analyse the $\mathcal{N}=1$ vacua. We summarize some of the properties here in the original bases.

$$\langle\!\langle \boldsymbol{D}_a \rangle\!\rangle \;=\; \delta_a{}^i \langle\!\langle \boldsymbol{D}_i \rangle\!\rangle = \delta_a{}^i O_i{}^{\underline{j}} \langle\!\langle \boldsymbol{D}_{\underline{j}} \rangle\!\rangle = \delta_a{}^i \sum_{\underline{j}} O_i{}^{\underline{j}} \boldsymbol{d}_{(\pm)} = \delta_a{}^i \delta_i{}^0 \sqrt{\frac{N}{2}} \boldsymbol{d}_{\pm} = \delta_a{}^0 \langle\!\langle \boldsymbol{D}_0 \rangle\!\rangle\ . \tag{4.11}$$

Recalling $\partial_a W = e\delta_a{}^0 + m\mathcal{F}_{a0}$, we see

$$\langle\!\langle \mathcal{F}_{a0} \rangle\!\rangle = \delta_a{}^0 \langle\!\langle \mathcal{F}_{00} \rangle\!\rangle\ , \qquad \langle\!\langle g_{a0} \rangle\!\rangle = \delta_a{}^0 \langle\!\langle g_{00} \rangle\!\rangle\ . \tag{4.12}$$

Hence

$$0 \;=\; \langle\!\langle \boldsymbol{D}_0^* \cdot \boldsymbol{D}_0^* \rangle\!\rangle = 2\langle\!\langle (\partial_0 W)^2 + \xi^2 \rangle\!\rangle = 2(e + m\langle\!\langle \mathcal{F}_{00} \rangle\!\rangle)^2 + 2\xi^2, \tag{4.13}$$

$$\langle\!\langle \mathcal{F}_{00} \rangle\!\rangle \;=\; -\left(\frac{e}{m} \pm i\frac{\xi}{m}\right), \tag{4.14}$$

$$\langle\!\langle \mathrm{Re}\mathcal{F}_{00} \rangle\!\rangle \;=\; -\frac{e}{m}, \tag{4.15}$$

$$\langle\!\langle g_{00} \rangle\!\rangle \;=\; \mp\frac{\xi}{m} = \left|\frac{\xi}{m}\right|. \tag{4.16}$$

After exhausting all possibilities, we conclude that partial spontaneous supersymmetry breaking takes place in the overall $U(1)$ sector. As for $\langle\!\langle \mathcal{V} \rangle\!\rangle$, we obtain

$$\langle\!\langle \mathcal{V} \rangle\!\rangle = \mp 2m\xi = 2|m\xi|. \tag{4.17}$$

V. Gauge symmetry breaking

Following the analysis of the vacua of our model, we turn to spontaneous breaking of gauge symmetry.

Let us recall that with the generic prepotential (3.12), the vacuum condition is given by (3.21):

$$\langle\!\langle \mathcal{F}_{\underline{i}\underline{i}} \rangle\!\rangle + 2\zeta = 0 \ , \tag{5.1}$$

with

$$\begin{aligned}
\langle\!\langle \mathcal{F}_{\underline{i}\underline{i}} \rangle\!\rangle &= \sum_{\ell}^{k=\deg \mathcal{F}} \frac{C_l}{(\ell-2)!} (\lambda^{\underline{i}})^{\ell-2} \\
&= C_2 + C_3 \lambda^{\underline{i}} + C_4 \frac{1}{2!} (\lambda^{\underline{i}})^2 + C_5 \frac{1}{3!} (\lambda^{\underline{i}})^3 + \cdots \ .
\end{aligned} \tag{5.2}$$

Here, we have introduced a complex parameter

$$\zeta \equiv \frac{e}{m} \pm i \frac{\xi}{m} \ . \tag{5.3}$$

Eq.(5.1) is an algebraic equation for $\lambda^{\underline{i}}$ with degree $\deg \mathcal{F} - 2 = k - 2$ and provides $k - 2$ complex roots denoted by $\lambda^{(\ell,\pm)}$, $\ell = 1 \sim k - 2$. Thus each $\lambda^{\underline{i}}$ is determined to be one of these $k - 2$ complex roots. As is well-known, this defines a grouping of N eigenvalues into $k-2$ sets and hence determines a breaking pattern of $U(N)$ gauge symmetry into a product gauge group $\prod_{i=1}^{k-2} U(N_i)$ with $\sum_{i=1}^{k-2} N_i = N$:

$$\langle\!\langle A \rangle\!\rangle = \begin{pmatrix}
\lambda^{(1,\pm)} & & & & & & & & \\
& \ddots & & & & & & & \\
& & \lambda^{(1,\pm)} & & & & & & \\
& & & \lambda^{(2,\pm)} & & & & & \\
& & & & \ddots & & & & \\
& & & & & \lambda^{(2,\pm)} & & & \\
& & & & & & \ddots & & \\
& & & & & & & \lambda^{(k-2,\pm)} & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \lambda^{(k-2,\pm)}
\end{pmatrix} . \tag{5.4}$$

In fact, as we will see in the next section, v_m^α are massless if $t_\alpha \in \{t_a \mid [t_a, \langle\!\langle A \rangle\!\rangle] = 0\}$ while v_m^μ are massive if $t_\mu \in \{t_a \mid [t_a, \langle\!\langle A \rangle\!\rangle] \neq 0\}$. The massless v_m^α contain gauge fields of unbroken $\prod_i U(N_i)$ and the superpartner of the Nambu-Goldstone fermion lies in the overall $U(1)$ part. We introduce

$$d_u \equiv \dim \prod_i U(N_i). \tag{5.5}$$

Let us, as a warm up, work out the case $N = 2$ and the case $N = 3$.

1) $N = 2$: there are two distinct cases for $\langle\!\langle A \rangle\!\rangle$

$$\text{i)} \begin{pmatrix} \lambda^{(\pm)} & 0 \\ 0 & \lambda^{(\pm)} \end{pmatrix}, \quad \text{ii)} \begin{pmatrix} \lambda^{(\pm)} & 0 \\ 0 & \lambda'^{(\pm)} \end{pmatrix} \text{ with } \lambda^{(\pm)} \neq \lambda'^{(\pm)}. \tag{5.6}$$

The diagonal entries of (5.6) are chosen from $\lambda^{(\ell,\pm)}$, $\ell = 1 \sim k-2$. In respective cases, $\mathcal{N} = 2$ supersymmetry and $U(2)$ gauge symmetry are broken respectively to i) $\mathcal{N} = 1$, $U(2)$ unbroken, ii) $\mathcal{N} = 1$, $U(1) \times U(1)$.

2) $N = 3$:

$$\text{i)} \begin{pmatrix} \lambda^{(\pm)} & 0 & 0 \\ 0 & \lambda^{(\pm)} & 0 \\ 0 & 0 & \lambda^{(\pm)} \end{pmatrix}, \quad \text{ii)} \begin{pmatrix} \lambda^{(\pm)} & 0 & 0 \\ 0 & \lambda^{(\pm)} & 0 \\ 0 & 0 & \lambda'^{(\pm)} \end{pmatrix} \text{ with } \lambda^{(\pm)} \neq \lambda'^{(\pm)},$$

$$\text{iii)} \begin{pmatrix} \lambda^{(\pm)} & 0 & 0 \\ 0 & \lambda'^{(\pm)} & 0 \\ 0 & 0 & \lambda''^{(\pm)} \end{pmatrix} \text{ with } \lambda^{(\pm)} \neq \lambda'^{(\pm)} \neq \lambda''^{(\pm)} \neq \lambda^{(\pm)},$$

and the unbroken symmetries are respectively i) $\mathcal{N} = 1$, $U(3)$ unbroken, ii) $\mathcal{N} = 1$, $U(2) \times U(1)$, iii) $\mathcal{N} = 1$, $U(1) \times U(1) \times U(1)$.

This pattern of symmetry breaking persists at general N. Let $m_\pm \equiv \mathrm{ord}(\boldsymbol{\mathcal{M}}_\pm)$, namely, the number of elements of $\boldsymbol{\mathcal{M}}_\pm$. Because either $m_+ = 0$ or $m_- = 0$, the unbroken symmetries are $\mathcal{N} = 1$, as is established in the last section, and the product gauge groups $\prod_{i=1}^{n} U(N_i)$ with $n \leq k-2$, N.

Let us finally discuss the condition under which zero eigenvalues of $\langle A \rangle$ emerge. This condition is simply

$$C_2 \pm 2\zeta = 0 \ , \tag{5.7}$$

as is read off from eqs.(5.1) and (5.2). The complex coefficients C_ℓ are to be determined from underlying microscopic theory and eq.(5.7) tells that we can always finetune the single complex parameter ζ to obtain the zero eigenvalues. When $\mathcal{F}_{\underline{ii}}$ is an even function of $\lambda^{\underline{i}}$, the condition means that eq.(5.1) has a double root. As a prototypical example, let $\mathcal{F}_{\underline{ii}}$ be even and $k = 6$. The roots are

$$\lambda^{\underline{i}} = \pm\sqrt{\frac{-\frac{C_4}{2} \pm \sqrt{\left(\frac{C_4}{2}\right)^2 - 4 \cdot \frac{C_6}{4!}(C_2 + 2\zeta)}}{2 \cdot \frac{C_6}{4!}}} \ . \tag{5.8}$$

When eq.(5.7) is satisfied, one of the two pairs of complex roots coalesces and indeed develops into zero consisting of a double root. This could be exploited to realize the "triplet-doublet splitting" at $N=5$ in the context of $SU(5)$ $\mathcal{N}=1$ SGUT. In the vacuum in which $U(5)$ is broken to $U(3)\times U(2)$ (or $SU(5)$ to $SU(3)\times SU(2)\times U(1)$, the standard model gauge group), we are able to obtain

$$\langle A\rangle = \begin{pmatrix} \lambda & & & & 0 \\ & \lambda & & & \\ & & \lambda & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix} , \tag{5.9}$$

in the case in which degeneration of eigenvalues is favored.

VI. Mass spectrum

We examine the mass spectrum for the $\mathcal{N}=1$ vacua for which $\forall \underline{i} \in \boldsymbol{\mathcal{M}}_{\pm}$.

A. Fermion mass spectrum

In this subsection, we compute the fermion masses. We examine the fermion mass term (2.7) for ψ^a and λ^a

$$-\frac{i}{4} g^{cd^*}\mathcal{F}_{abc}\partial_{d^*}W^*(\psi^a\psi^b+\lambda^a\lambda^b)+\frac{1}{2\sqrt{2}}\left(g_{ac^*}k_b^{*c}-g_{bc^*}k_a^{*c}-\sqrt{2}\xi\delta_c^0 g^{cd}\mathcal{F}_{abd}\right)\psi^a\lambda^b \ . \tag{6.1}$$

The first term in (6.1) becomes

$$-\frac{i}{4}\langle\!\langle g^{kl}\mathcal{F}_{ijk}\partial_{l^*}W^*\rangle\!\rangle(\psi^i\psi^j+\lambda^i\lambda^j)-\frac{i}{4}\langle\!\langle g^{ij}\mathcal{F}_{rsi}\partial_{j^*}W^*\rangle\!\rangle(\psi^r\psi^s+\lambda^r\lambda^s) \ , \tag{6.2}$$

because $\langle\!\langle \partial_{a^*}W^*\rangle\!\rangle=\delta_{a^*}^{i^*}\langle\!\langle\partial_{i^*}W^*\rangle\!\rangle$ and $\langle\!\langle\mathcal{F}_{ijr}\rangle\!\rangle=0$. The second term in (6.1) reduces to

$$-\frac{1}{2}\langle\!\langle\xi g^{0k}\mathcal{F}_{kij}\rangle\!\rangle\psi^i\lambda^j+\frac{1}{2\sqrt{2}}\langle\!\langle g_{st}f_{ri}^t A^{*i}\rangle\!\rangle(\psi^s\lambda^r-\psi^r\lambda^s)-\frac{1}{2}\langle\!\langle\xi g^{0i}\mathcal{F}_{irs}\rangle\!\rangle\psi^r\lambda^s, \tag{6.3}$$

as $\langle\!\langle g_{ac^*}k_b^{*c}\rangle\!\rangle\psi^a\lambda^b=\langle\!\langle g_{ac^*}f_{ri}^c A^{*i}\rangle\!\rangle\psi^a\lambda^r=\langle\!\langle g_{st}f_{ri}^t A^{*i}\rangle\!\rangle\psi^s\lambda^r$. We have used $f_{ri}^c\langle\!\langle A^{*i}\rangle\!\rangle=f_{ri}^t\delta_t^c\langle\!\langle A^{*i}\rangle\!\rangle$ in the last equality. We thus conclude that the mass terms of the fermions with i index are decoupled from those of the fermions with r index.

From (6.2) and (6.3), the mass term for the $\boldsymbol{\lambda}_I^{\underline{i}}$ in the eigenvalue basis is written as $\frac{1}{2}\boldsymbol{\lambda}^{\underline{i}I}(M_{\underline{ii}})_I{}^J\boldsymbol{\lambda}_J^{\underline{i}}$ with

$$(M_{\underline{ii}})_I{}^J=-\frac{i}{2}\langle\!\langle g^{\underline{ii}}\mathcal{F}_{\underline{iii}}\rangle\!\rangle\begin{pmatrix} -i\xi O_{\underline{i}}^0 & \langle\!\langle\partial_{\underline{i}^*}W^*\rangle\!\rangle \\ -\langle\!\langle\partial_{\underline{i}^*}W^*\rangle\!\rangle & +i\xi O_{\underline{i}}^0\end{pmatrix}=\frac{1}{2\sqrt{2}}\langle\!\langle\mathcal{F}_{\underline{iii}}\rangle\!\rangle(\boldsymbol{\tau}\cdot\langle\!\langle\boldsymbol{D}^{\underline{i}}\rangle\!\rangle) \ . \tag{6.4}$$

It is easy to show that the mass term becomes of the form

$$\frac{i}{8\sqrt{2}}\sum_{\underline{i}}\langle\!\langle D_2^{\underline{i}}+iD_3^{\underline{i}}\rangle\!\rangle\langle\!\langle\mathcal{F}_{\underline{iii}}\rangle\!\rangle(\lambda^{\underline{i}}+\psi^{\underline{i}})^2+\frac{i}{8\sqrt{2}}\sum_{\underline{i}}\langle\!\langle D_2^{\underline{i}}-iD_3^{\underline{i}}\rangle\!\rangle\langle\!\langle\mathcal{F}_{\underline{iii}}\rangle\!\rangle(\lambda^{\underline{i}}-\psi^{\underline{i}})^2\ . \tag{6.5}$$

Noting that for the $\mathcal{N}=1$ vacua, $\forall\underline{i}\in\boldsymbol{\mathcal{M}}_\pm$,

$$\langle\!\langle D_2^{\underline{i}}\mp iD_3^{\underline{i}}\rangle\!\rangle=2i\frac{m}{\sqrt{N}}\ ,\quad \langle\!\langle D_2^{\underline{i}}\pm iD_3^{\underline{i}}\rangle\!\rangle=0\ , \tag{6.6}$$

we find that fermions $\frac{1}{\sqrt{2}}(\lambda^{\underline{i}}\pm\psi^{\underline{i}})$ are massless while fermions $\frac{1}{\sqrt{2}}(\lambda^{\underline{i}}\mp\psi^{\underline{i}})$ are massive with mass $|m\langle\!\langle g^{\underline{ii}}\rangle\!\rangle\langle\!\langle\mathcal{F}_{0\underline{ii}}\rangle\!\rangle|$ after using (3.11). Here $g^{\underline{ii}}$ comes from the normalization of the kinetic term.

On the other hand, it follows from (6.2) and (6.3) that the mass term of $\boldsymbol{\lambda}_I^r$ is written as $\frac{1}{2}\boldsymbol{\lambda}^{rI}(M_{rs})_I{}^J\boldsymbol{\lambda}_J^s$ with

$$(M_{rs})_I{}^J=\begin{pmatrix}\mp m_{rs}+m'_{rs} & m_{rs}\\ -m_{rs} & \pm m_{rs}+m'_{rs}\end{pmatrix}\ , \tag{6.7}$$

$$m_{rs}=-\frac{1}{2}m\langle\!\langle\mathcal{F}_{0rs}\rangle\!\rangle\ ,\quad m'_{rs}=\frac{1}{2\sqrt{2}}\left(\langle\!\langle g_{tr}\rangle\!\rangle f^t_{s\underline{i}}\lambda^{*\underline{i}}-\langle\!\langle g_{ts}\rangle\!\rangle f^t_{r\underline{i}}\lambda^{*\underline{i}}\right)\ . \tag{6.8}$$

It is convenient to express the index r as a union of the two indices r' and μ such that $t_{r'}\in\{t_r\mid[t_r,\langle\!\langle A\rangle\!\rangle]=0\}$ and $t_\mu\in\{t_r\mid[t_r,\langle\!\langle A\rangle\!\rangle]\neq0\}$. Since $[t_\mu,\langle\!\langle A\rangle\!\rangle]$ belongs to $\{t_\mu\}$ and $\langle\!\langle g_{st}\rangle\!\rangle$ is diagonal, m'_{rs} is nonvanishing only for $m'_{\mu\nu}$. On the other hand, m_{rs} is nonvanishing only for $m_{r's'}$, as $\langle\!\langle\mathcal{F}_{0\mu\nu}\rangle\!\rangle=0$. This last equality is proven from (3.17) and (5.1) by

$$\langle\!\langle\mathcal{F}_{0,\pm\underline{ij},\pm\underline{ij}}\rangle\!\rangle=\frac{\langle\!\langle\mathcal{F}_{\underline{ii}}\rangle\!\rangle-\langle\!\langle\mathcal{F}_{\underline{jj}}\rangle\!\rangle}{2\sqrt{2N}(\lambda^{\underline{i}}-\lambda^{\underline{j}})}=\frac{-2\zeta-(-2\zeta)}{2\sqrt{2N}(\lambda^{\underline{i}}-\lambda^{\underline{j}})}=0\ . \tag{6.9}$$

Thus, we find that the $\boldsymbol{\lambda}_I^{r'}$ mass term decouples from the $\boldsymbol{\lambda}_I^{\mu}$ mass term. The $\boldsymbol{\lambda}_I^{r'}$ mass term can be easily diagonalized as was done for $\boldsymbol{\lambda}_I^{\underline{i}}$, and we find that $\frac{1}{\sqrt{2}}(\lambda^{r'}\pm\psi^{r'})$ are massless and $\frac{1}{\sqrt{2}}(\lambda^{r'}\mp\psi^{r'})$ are massive with mass $|m\langle\!\langle g^{r'r'}\rangle\!\rangle\langle\!\langle\mathcal{F}_{0r'r'}\rangle\!\rangle|$. Combining the result on $\boldsymbol{\lambda}_I^{\underline{i}}$ with that on $\boldsymbol{\lambda}_I^{r'}$ and letting $\alpha=\underline{i}\cup r'$ namely $t_\alpha\in\{t_a\mid[t_a,\langle\!\langle A\rangle\!\rangle]=0\}$, we find that $\frac{1}{\sqrt{2}}(\lambda^\alpha\pm\psi^\alpha)$ are massless while $\frac{1}{\sqrt{2}}(\lambda^\alpha\mp\psi^\alpha)$ are massive with mass $|m\langle\!\langle g^{\alpha\alpha}\rangle\!\rangle\langle\!\langle\mathcal{F}_{0\alpha\alpha}\rangle\!\rangle|$.

Finally, we examine the mass of $\boldsymbol{\lambda}_I^\mu$. Denoting $E_\pm^{\underline{ij}}$ with $\lambda^{\underline{i}}\neq\lambda^{\underline{j}}$ by $t_{\mu_\pm}$ for short, we find that $m'_{\mu\nu}$ is nonvanishing only for $m'_{\mu_+\mu_-}=-m'_{\mu_-\mu_+}$, since $[\langle\!\langle A\rangle\!\rangle,t_{\mu_\pm}]\propto t_{\mu_\mp}$. (See the Appendix.) In these indices, the $\boldsymbol{\lambda}_I^{\mu_\pm}$ mass term is written as

$$\frac{1}{2}\boldsymbol{\lambda}^{\mu_+I}\begin{pmatrix}2m'_{\mu_+\mu_-} & 0\\ 0 & 2m'_{\mu_+\mu_-}\end{pmatrix}_I{}^J\boldsymbol{\lambda}_J^{\mu_-}. \tag{6.10}$$

The summation is implied only for either one of the two indices μ_+ or μ_-, and it is over half of the broken generators. This implies that $\boldsymbol{\lambda}_I^\mu$ has mass $\left|g^{\nu\nu}m'_{\mu\nu}\right|=\left|g^{\mu_-\mu_-}m'_{\mu_+\mu_-}\right|=\frac{1}{\sqrt{2}}\left|f^\nu_{\mu\underline{i}}\lambda^{*\underline{i}}\right|$ where we have used the fact $\langle\!\langle g_{\mu_+\mu_+}\rangle\!\rangle=\langle\!\langle g_{\mu_-\mu_-}\rangle\!\rangle$ and $f^{\mu_-}_{\mu_+\underline{i}}\lambda^{*\underline{i}}=-f^{\mu_+}_{\mu_-\underline{i}}\lambda^{*\underline{i}}$.

B. Boson mass spectrum

In order to obtain the boson masses of our model, we evaluate the second variation of the scalar potential on the vacuum $\langle\!\langle \delta\delta\mathcal{V} \rangle\!\rangle$. Recall that

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2, \tag{6.11}$$

$$\mathcal{V}_1 = \frac{1}{8} g_{ab} \mathfrak{D}^a \mathfrak{D}^b, \quad \mathfrak{D}^a = -i f^a_{cd} A^{*c} A^d, \tag{6.12}$$

$$\begin{aligned} \mathcal{V}_2 &= g^{ab} \left(\xi^2 \delta_a{}^0 \delta_b{}^0 + \partial_a W \partial_b W^* \right) \\ &= (\xi^2 + e^2) g^{00} + m^2 g_{00} + 2me(\mathrm{Re}\mathcal{F}_{0b}) g^{b0} + m^2 (\mathrm{Re}\mathcal{F}_{0b}) g^{bc} \mathrm{Re}\mathcal{F}_{0c} \ . \end{aligned} \tag{6.13}$$

Let us first compute $\langle\!\langle \delta\delta\mathcal{V}_1 \rangle\!\rangle$. From $\langle\!\langle \mathfrak{D}^a \rangle\!\rangle = 0$ and $\langle\!\langle A^a \rangle\!\rangle = \delta^a{}_{\underline{j}} \langle\!\langle A^{\underline{j}} \rangle\!\rangle$, we obtain

$$\langle\!\langle \delta\mathfrak{D}^a \rangle\!\rangle = \delta^a{}_{\tilde{\mu}} \langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle \ , \tag{6.14}$$

$$\langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle = \langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle^* = \left((-i) f^{\tilde{\mu}}_{\underline{j}\mu} \lambda^{*\underline{j}}, \ (+i) f^{\tilde{\mu}}_{\underline{j}\mu} \lambda^{\underline{j}} \right) \begin{pmatrix} \delta A^\mu \\ \delta A^{*\mu} \end{pmatrix} \equiv \left(\overrightarrow{(f_\perp \lambda^*)^{\tilde{\mu}}{}_\mu} \right)^t \cdot \overrightarrow{\delta A^\mu} \ . \tag{6.15}$$

Here $f^{\tilde{\mu}}_{\underline{j}\mu}$ is the structure constant of the $u(N)$ Lie algebra which is read off from the Appendix. For given μ specified by a pair of indices $(\underline{k}, \underline{l})$, $1 \le \underline{k} \le \underline{l} \le N$, $\tilde{\mu}$ is uniquely determined and vice versa and the summation over $\underline{j}$ reduces to that of $\underline{j} = \underline{k}$ and $\underline{j} = \underline{l}$. We obtain

$$\langle\!\langle \delta\delta\mathcal{V}_1 \rangle\!\rangle = \frac{1}{4} \sum_{\tilde{\mu}} \langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle \langle\!\langle g_{\tilde{\mu}\tilde{\mu}} \rangle\!\rangle \langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle. \tag{6.16}$$

The summation over $\tilde{\mu}$ is for $N^2 - d_u$ directions of the broken generators.

In order to separate the $N^2 - d_u$ Nambu-Goldstone zero modes (one mode for each μ), we introduce the following projector of a 2×2 matrix as a diad for each μ:

$$\mathcal{P}^{\tilde{\mu}}{}_\mu \equiv \frac{1}{||\overrightarrow{(f_\perp \lambda)^{\tilde{\mu}}{}_\mu}||^2} \overrightarrow{(f_\perp \lambda)^{\tilde{\mu}}{}_\mu} \left(\overrightarrow{(f_\perp \lambda^*)^{\tilde{\mu}}{}_\mu} \right)^t, \tag{6.17}$$

where

$$\begin{aligned} ||\overrightarrow{(f_\perp \lambda)^{\tilde{\mu}}{}_\mu}||^2 &\equiv \left(\overrightarrow{(f_\perp \lambda^*)^{\tilde{\mu}}{}_\mu} \right)^t \cdot \overrightarrow{(f_\perp \lambda)^{\tilde{\mu}}{}_\mu} \\ &= 2 |f^{\tilde{\mu}}_{\underline{j}\mu} \lambda^{\underline{j}}|^2. \end{aligned} \tag{6.18}$$

We express $\langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle$ as

$$\langle\!\langle \delta\mathfrak{D}^{\tilde{\mu}} \rangle\!\rangle = \left(\overrightarrow{(f_\perp \lambda^*)^{\tilde{\mu}}{}_\mu} \right)^t \cdot \left(\mathcal{P}^{\tilde{\mu}}{}_\mu \overrightarrow{\delta A^\mu} \right) \ . \tag{6.19}$$

The mode orthogonal to $\left(\mathcal{P}^{\tilde{\mu}}{}_{\mu}\overrightarrow{\delta A^{\mu}}\right)$ is the one belonging to the zero eigenvalue for each μ in eq.(6.16) and is absorbed into the longitudinal components of the corresponding gauge fields. It is given by

$$(\mathbf{1}_2 - \mathcal{P}^{\tilde{\mu}}{}_{\mu})\overrightarrow{\delta A^{\mu}}, \tag{6.20}$$

with

$$\mathbf{1}_2 - \mathcal{P}^{\tilde{\mu}}{}_{\mu} = \frac{1}{||\overrightarrow{(f_\perp\lambda)^{\tilde{\mu}}_{\mu}}||^2}\begin{pmatrix} f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{\underline{j}} \\ f^{\tilde{\bar{\mu}}}_{\underline{j}\mu}\lambda^{*\underline{j}} \end{pmatrix}\left(f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{*\underline{j}},\ f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{\underline{j}}\right) . \tag{6.21}$$

Substituting (6.19) into (6.16), we obtain

$$\langle\!\langle \delta\delta\mathcal{V}_1 \rangle\!\rangle = \frac{1}{4}\sum_{\tilde{\mu}} \langle\!\langle g_{\tilde{\mu}\tilde{\mu}}\rangle\!\rangle |f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{\underline{j}}|^2 \left(\mathcal{P}^{\tilde{\mu}}{}_{\mu}\overrightarrow{\delta A^{\mu}}\right)^{*t}\cdot\left(\mathcal{P}^{\tilde{\mu}}{}_{\mu}\overrightarrow{\delta A^{\mu}}\right) . \tag{6.22}$$

Let us turn our attention to $\langle\!\langle \delta\delta\mathcal{V}_2 \rangle\!\rangle$.

$$\begin{aligned} \delta\mathcal{V}_2 &= -(\xi^2+e^2)(g^{-1}\delta g g^{-1})^{00} + m^2(\delta g)_{00} \\ &\quad +2me(\delta\mathrm{Re}\mathcal{F}_{0b})g^{b0} - 2me(\mathrm{Re}\mathcal{F}_{0b})(g^{-1}\delta g g^{-1})^{b0} \\ &\quad +2m^2(\delta\mathrm{Re}\mathcal{F}_{0b})g^{bc}\mathrm{Re}\mathcal{F}_{0c} - m^2(\mathrm{Re}\mathcal{F}_{0b})(g^{-1}\delta g g^{-1})^{bc}(\mathrm{Re}\mathcal{F}_{0c}) . \end{aligned} \tag{6.23}$$

It is easy to check $\langle\!\langle \delta\mathcal{V}_2 \rangle\!\rangle = 0$, as it should be, from the properties (4.14)-(4.16) listed in the end of section IV. It is straightforward to carry out one more variation, which we will not spell out here. Again from (4.14)-(4.16), we obtain

$$\begin{aligned} \langle\!\langle \delta\delta\mathcal{V}_2 \rangle\!\rangle &= 2m^2 \langle\!\langle \delta\mathcal{F}^*_{0c}\rangle\!\rangle \langle\!\langle g^{cd}\rangle\!\rangle \langle\!\langle \delta\mathcal{F}_{0d}\rangle\!\rangle \\ &= 2m^2 \delta A^{*b}\langle\!\langle \mathcal{F}^*_{0bc}\rangle\!\rangle \langle\!\langle g^{cd}\rangle\!\rangle \langle\!\langle \mathcal{F}_{0ad}\rangle\!\rangle \delta A^a \\ &= 2m^2 \delta A^{*\alpha}\langle\!\langle \mathcal{F}^*_{0\alpha\alpha}\rangle\!\rangle \langle\!\langle g^{\alpha\alpha}\rangle\!\rangle \langle\!\langle \mathcal{F}_{0\alpha\alpha}\rangle\!\rangle \delta A^\alpha . \end{aligned} \tag{6.24}$$

In deriving the last line, we have used eq.(6.9) derived in the preceding subsection.

Combining the two calculations (6.22) and (6.24), we can read off the mass formula of the scalar bosons from $\langle\!\langle \delta\delta\mathcal{V} \rangle\!\rangle = \langle\!\langle \delta\delta\mathcal{V}_1 \rangle\!\rangle + \langle\!\langle \delta\delta\mathcal{V}_2 \rangle\!\rangle$. The scalar masses arising from the directions of the unbroken generators are given by $|m\langle\!\langle g^{\alpha\alpha}\rangle\!\rangle\langle\!\langle \mathcal{F}_{0\alpha\alpha}\rangle\!\rangle|$. Here, we have interpreted $\langle\!\langle g^{\alpha\alpha}\rangle\!\rangle$ as a factor due to the normalization of the kinetic term. The scalar masses arising from the directions of the broken generators are given by $\frac{1}{\sqrt{2}}|f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{\underline{j}}|$, after the normalization we just discussed. Finally we read off the mass of the massive gauge bosons from $-\langle\!\langle \mathcal{L}_{\mathrm{kin}} \rangle\!\rangle$.

$$\begin{aligned} -\langle\!\langle \mathcal{L}_{\mathrm{kin}} \rangle\!\rangle &= \langle\!\langle g_{aa'}\rangle\!\rangle \frac{1}{2} f^a_{bc} v_m{}^b \langle\!\langle A^c \rangle\!\rangle \frac{1}{2} f^{a'}_{b'c'} v^{mb'} \langle\!\langle A^{c'} \rangle\!\rangle \\ &= \frac{1}{4}\sum_{\mu} |f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{\underline{j}}|^2 v_m{}^{\mu} v^{m\mu}. \end{aligned} \tag{6.25}$$

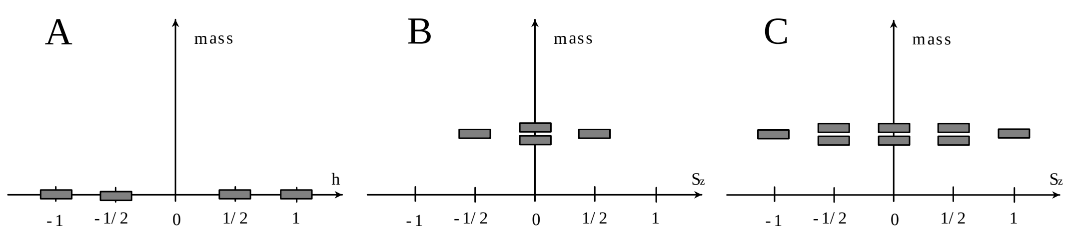
The mass is given by $\frac{1}{\sqrt{2}}|f^{\tilde{\mu}}_{\underline{j}\mu}\lambda^{\underline{j}}|$.

We summarize the spectrum of the bosons and the fermions by the following table:

field	mass	label	# of polarization states
v^{α}_{m}	0	A	$2d_u$
v^{μ}_{m}	$\frac{1}{\sqrt{2}}\|f^{\nu}_{\mu\underline{i}}\lambda^{\underline{i}}\|$	C	$3(N^2-d_u)$
$\frac{1}{\sqrt{2}}(\lambda^{\alpha}\pm\psi^{\alpha})$	0	A	$2d_u$
$\frac{1}{\sqrt{2}}(\lambda^{\alpha}\mp\psi^{\alpha})$	$\|m\langle\langle g^{\alpha\alpha}\rangle\rangle\langle\langle\mathcal{F}_{0\alpha\alpha}\rangle\rangle\|$	B	$2d_u$
$\boldsymbol{\lambda}^{\mu}_{I}$	$\frac{1}{\sqrt{2}}\|f^{\nu}_{\mu\underline{i}}\lambda^{\underline{i}}\|$	C	$4(N^2-d_u)$
A^{α}	$\|m\langle\langle g^{\alpha\alpha}\rangle\rangle\langle\langle\mathcal{F}_{0\alpha\alpha}\rangle\rangle\|$	B	$2d_u$
$\mathcal{P}^{\tilde{\mu}}_{\mu}A^{\mu}$	$\frac{1}{\sqrt{2}}\|f^{\nu}_{\mu\underline{i}}\lambda^{\underline{i}}\|$	C	N^2-d_u

We find the following supermultiplets. First, we consider the massless particles associated with $\frac{1}{\sqrt{2}}(\lambda^{\alpha}\pm\psi^{\alpha})$ and v^{α}_{m}. These are labelled as A in the table. These form massless $\mathcal{N}=1$ vector multiplets of spin $(1/2,1)$, in which the Nambu-Goldstone vector multiplet is contained in the overall $U(1)$ part. Second, massive particles labelled as B, which are associated with $\frac{1}{\sqrt{2}}(\lambda^{\alpha}\mp\psi^{\alpha})$ and A^{α}, have masses given by $|m\langle\langle g^{\alpha\alpha}\rangle\rangle\langle\langle\mathcal{F}_{0\alpha\alpha}\rangle\rangle|$. These form massive $\mathcal{N}=1$ chiral multiplets of spin $(0,1/2)$. Finally, we consider massive particles labelled as C. The zero modes of A^{μ} are absorbed into v^{μ}_{m} as the longitudinal modes to form massive vector fields. These form two massive multiplets of spin $(0,1/2,1)$ with the massive modes of A^{μ} and $\boldsymbol{\lambda}^{\mu}_{I}$. The masses of these supermultiplets are given by $\frac{1}{\sqrt{2}}\left|f^{\nu}_{\mu\underline{i}}\lambda^{\underline{i}}\right|$.

In the following figure, the masses of the three types of $\mathcal{N}=1$ supermultiplets are schematically drawn.



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Appendix

Let $E_{\underline{ij}}$, $\underline{i},\underline{j}=\underline{1},...,\underline{N}$, be the fundamental matrix, which has 1 at the $(\underline{i},\underline{j})$-component and 0 otherwise. Cartan generators of $u(N)$ are $H_{\underline{i}}\equiv E_{\underline{ii}}$, which are denoted as $t_{\underline{i}}$ in section 3. Non-Cartan generators can be written as

$$E^{+}_{\underline{ij}}=E^{+}_{\underline{ji}}\equiv\frac{1}{2}(E_{\underline{ij}}+E_{\underline{ji}})\ ,\quad E^{-}_{\underline{ij}}=-E^{-}_{\underline{ji}}\equiv-\frac{i}{2}(E_{\underline{ij}}-E_{\underline{ji}})\ ,\qquad \underline{i}\neq\underline{j}\ , \tag{A.1}$$

which are normalized as $\mathrm{tr}(E^{\pm}_{\underline{ij}})^2=\frac{1}{2}$. Commutation relations are

$$\begin{aligned}
[H_{\underline{i}},E^{\pm}_{\underline{jk}}] &= \pm i\delta_{\underline{ij}}E^{\mp}_{\underline{ik}}+i\delta_{\underline{ik}}E^{\mp}_{\underline{ij}}\ , && (A.2)\\
[E^{\pm}_{\underline{ij}},E^{\pm}_{\underline{kl}}] &= \pm 2i\delta_{\underline{jk}}E^{-}_{\underline{il}}+2i\delta_{\underline{ik}}E^{-}_{\underline{jl}}+2i\delta_{\underline{jl}}E^{-}_{\underline{ik}}\pm 2i\delta_{\underline{il}}E^{-}_{\underline{jk}}\ , && (A.3)\\
[E^{+}_{\underline{ij}},E^{-}_{\underline{kl}}] &= -2i\delta_{\underline{jk}}(E^{+}_{\underline{il}}+\delta_{\underline{il}}H_{\underline{i}})-2i\delta_{\underline{ik}}(E^{+}_{\underline{jl}}+\delta_{\underline{jl}}H_{\underline{j}})\\
&\quad +2i\delta_{\underline{jl}}(E^{+}_{\underline{ik}}+\delta_{\underline{ik}}H_{\underline{i}})+2i\delta_{\underline{il}}(E^{+}_{\underline{jk}}+\delta_{\underline{jk}}H_{\underline{j}})\ . && (A.4)
\end{aligned}$$

By introducing the vacuum expectation value $\langle\Phi\rangle=\lambda^{\underline{i}}t_{\underline{i}}$, it follows that

$$[E^{\pm}_{\underline{jk}},\langle A\rangle] = [E^{\pm}_{\underline{jk}},\lambda^{\underline{i}}H_{\underline{i}}]=\mp i(\lambda^{\underline{j}}-\lambda^{\underline{k}})E^{\mp}_{\underline{jk}}\ . \tag{A.5}$$

In the text, $E^{\pm}_{\underline{jk}}$ are denoted as $t_{r'}$ when $\lambda^{\underline{j}}=\lambda^{\underline{k}}$ while t_μ when $\lambda^{\underline{j}}\neq\lambda^{\underline{k}}$. Explicitly, $\Phi=\Phi^a t_a$ is expanded as follows:

$$\begin{aligned}
\Phi &= \Phi^i t_i+\Phi^r t_r\ ,\\
\Phi^i t_i &= \Phi^{\underline{i}}t_{\underline{i}}\ ,\quad \Phi^r t_r=\Phi^{r'}t_{r'}+\Phi^\mu t_\mu\ ,\\
\Phi^{r'}t_{r'} &= \left[\frac{1}{2}\Phi^{\underline{ij}}_{+}E^{+}_{\underline{ij}}+\frac{1}{2}\Phi^{\underline{ij}}_{-}E^{-}_{\underline{ij}}\right]_{\lambda^{\underline{i}}=\lambda^{\underline{j}}}\ ,\quad \Phi^\mu t_\mu=\left[\frac{1}{2}\Phi^{\underline{ij}}_{+}E^{+}_{\underline{ij}}+\frac{1}{2}\Phi^{\underline{ij}}_{-}E^{-}_{\underline{ij}}\right]_{\lambda^{\underline{i}}\neq\lambda^{\underline{j}}}\ .
\end{aligned} \tag{A.6}$$

Let $\mathcal{F}$ be $\mathcal{F} = \sum_\ell \frac{C_\ell}{\ell!} \operatorname{tr} \Phi^\ell$ then $\langle \mathcal{F}_{ab} \rangle$ are evaluated as

$$\begin{aligned}
\langle \mathcal{F}_{\underline{i}\underline{j}} \rangle &= \sum_\ell \frac{C_\ell}{(\ell-1)!} \sum_{l'=0}^{\ell-2} \operatorname{tr}(H_{\underline{i}} \langle A \rangle^{l} H_{\underline{j}} \langle A \rangle^{\ell-2-l'}) \\
&= \sum_\ell \frac{C_\ell}{(\ell-1)!} \sum_{l'=0}^{\ell-2} \operatorname{tr}(H_{\underline{i}} H_{\underline{i}'} H_{\underline{j}} H_{\underline{j}'}) (\lambda^{\underline{i}'})^{l'} (\lambda^{\underline{j}'})^{\ell-2-l'} \\
&= \sum_\ell \frac{C_\ell}{(\ell-2)!} \delta_{\underline{i}\underline{j}} (\lambda^{\underline{i}})^{\ell-2} \, , \qquad \text{(A.7)}
\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{F}_{\pm\underline{i}\underline{j}, \pm\underline{i}\underline{j}} \rangle &= \langle \frac{\partial^2 \mathcal{F}}{\partial \Phi_\pm^{\underline{i}\underline{j}} \partial \Phi_\pm^{\underline{i}\underline{j}}} \rangle = \sum_\ell \frac{C_\ell}{(\ell-1)!} \sum_{l'=0}^{\ell-2} \operatorname{tr}(E^\pm_{\underline{i}\underline{j}} H_{\underline{i}'} E^\pm_{\underline{i}\underline{j}} H_{\underline{j}'}) (\lambda^{\underline{i}'})^{l'} (\lambda^{\underline{j}'})^{\ell-2-l'} \\
&= \begin{cases} \displaystyle\sum_\ell \frac{C_\ell}{2(\ell-1)!} \frac{(\lambda^{\underline{i}})^{\ell-1} - (\lambda^{\underline{j}})^{\ell-1}}{\lambda^{\underline{i}} - \lambda^{\underline{j}}} & (\lambda^{\underline{i}} \neq \lambda^{\underline{j}} : \text{broken}) \\ \displaystyle\sum_\ell \frac{C_\ell}{2(\ell-2)!} (\lambda^{\underline{i}})^{\ell-2} & (\lambda^{\underline{i}} = \lambda^{\underline{j}} : \text{unbroken}) \end{cases} \qquad \text{(A.8)}
\end{aligned}$$

and the others vanish. We have used (A.5) and

$$E^\pm_{\underline{i}\underline{j}} E^\pm_{\underline{k}\underline{l}} = \frac{1}{4} (\delta_{\underline{i}\underline{k}} \delta_{\underline{j}\underline{l}} \pm \delta_{\underline{i}\underline{l}} \delta_{\underline{j}\underline{k}}) (H_{\underline{i}} + H_{\underline{j}}) + \text{non-Cartan generators} \, , \qquad \text{(A.9)}$$

$$E^\pm_{\underline{i}\underline{j}} E^\mp_{\underline{k}\underline{l}} = \pm \frac{i}{4} (\delta_{\underline{i}\underline{k}} \delta_{\underline{j}\underline{l}} \mp \delta_{\underline{i}\underline{l}} \delta_{\underline{j}\underline{k}}) (H_{\underline{i}} - H_{\underline{j}}) + \text{non-Cartan generators} \qquad \text{(A.10)}$$

in the calculation.

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